Assignment 7

1. Apply two steps of Newton's method to find a simultaneous root of the following system of two algebraic equations starting with the approximation $x = y = 0.5$.

$$x^2 + 4y^2 - x + y - 1 = 0$$
$$3x^2 + 2y^2 + x - y - 2 = 0$$

The Jacobian is the matrix with the first row being $(2x - 1, 8y + 1)$ and the second row $(6x + 1, 4y - 1)$.

In MATLAB, we can code the function and the Jacobian as follows; although you can do it by hand, too,

```
>> f = @(u)([ u(1)^2 + 4*u(2)^2 - u(1) + u(2) - 1
              3*u(1)^2 + 2*u(2)^2 + u(1) - u(2) - 2 ]);
>> J = @(u)([ 2*u(1) - 1, 8*u(2) + 1
              6*u(1) + 1, 4*u(2) - 1 ]);
>> u0 = [0.5; 0.5];
>> f(u0)
       0.25
      -0.75

>> % First iteration
>> du0 = J(u0) \ -f(u0)
    du0 =
       0.2
      -0.05
>> u1 = u0 + du0
    u1 =
       0.7
       0.45
>> f(u1)     % This should be smaller
       0.05
       0.125

>> % Second iteration
>> du1 = J(u1) \ -f(u1)
    du1 =
      -0.02266949152542365
      -0.008898305084745780
>> u2 = u1 + du1
    u2 =
       0.6773305084745763
       0.4411016949152542
>> f( u2 )   % This should be even smaller
    ans =
       0.0008306251795462405
       0.001700077204826567
```

2. Apply one step of Newton's method to find a simultaneous root of the following system of three algebraic equations starting with the approximation $x = y = -0.5$ and $z = 1$.

$$3x^2 + x - xy - 2y - 1 = 0$$
$$2x + 2y^2 + xy - y + z - yz - 2 = 0$$
$$y - 2z + yz + 3z^2 = 0$$

The Jacobian is the matrix with the first row being $(6x + 1 - y, -x - 2, 0)$, the second $(2 + y, 4y + x - 1 - z, 1 - y)$, and the third $(0, 1 + z, -2 + y + 6z)$.

Two iterations are performed for interest:

```
f = @(u)([ 3*u(1)^2 + u(1) - u(1)*u(2) - 2*u(2) - 1
           2*u(1) + 2*u(2)^2 + u(1)*u(2) - u(2) + u(3) - u(2)*u(3) - 2
           u(2) - 2*u(3) + u(2)*u(3) + 3*u(3)^2 ]);
J = @(u)([ 6*u(1) + 1 - u(2), -u(1) - 2,                 0
           2 + u(2),           4*u(2) + u(1) - 1 - u(3),  1 - u(2)
           0,                  1 + u(3),                 -2 + u(2) + 6*u(3)]);
u0 = [-0.5; -0.5; 1];
f( u0 )
   0
  -0.25
   0

% First iteration
du0 = J(u0) \ -f(u0)
    du0 =
      0.03645833333333333
     -0.03645833333333333
      0.02083333333333333

u1 = u0 + du0
    u1 =
     -0.4635416666666667
     -0.5364583333333334
      1.020833333333333e

f( u1 )
      0.005316840277777901
      0.002088758680555358
      0.0005425347222214327


% Second iteration
du1 = J(u1) \ -f(u1)
    du1 =
      0.002900757102792720
      0.001110347070392873
     -0.0007764605660071837

u2 = u1 + du1
    u2 =
      -0.4606409095638740
      -0.5353479862629404
       1.020056872767326

f( u2 )
       0.000002202232815706751
       0.0000006548729099442596
       0.00000009465323174140394
```

3. Apply one step of Newton's method to find a simultaneous root of the following system of two algebraic equations starting with the approximation $x = 1$ and $y = 1.5$.

$$\sin(x) + 2\cos(xy) = 1$$
$$\sin(xy) - 2\cos(y) = 1$$

The Jacobian is the matrix with the first row being $(\cos(x) - 2\sin(xy)y, -2\sin(xy)x)$ and the second $(\cos(xy)y, \cos(xy)x + 2\sin(y))$.

Two iterations are performed for interest:

```
f = @(u)([ sin(u(1))      + 2*cos(u(1)*u(2)) - 1
           sin(u(1)*u(2)) - 2*cos(u(2)) - 1 ]);
J = @(u)([ cos(u(1)) - 2*sin(u(1)*u(2))*u(2), -2*sin(u(1)*u(2))*u(1)
           cos(u(1)*u(2))*u(2), cos(u(1)*u(2))*u(1) + 2*sin(u(2)) ]);
u0 = [1.0; 1.5];
f( u0 )
      -0.01705461185669765
      -0.1439794167313514

% First iteration
du0 = J(u0) \ -f(u0)
    du0 =
      -0.06643530009365024
       0.07311158482063311
u1 = u0 + du0
    u1 =
       0.9335646999063497
       1.573111584820633
f( u1 )
       0.007780124676444178
      -0.0005868419391285018

du1 = J(u1) \ -f(u1)
    du1 =
       0.003034353204493463
       0.00004766817273407990
u2 = u1 + du1
    u2 =
       0.9365990531108431
       1.573159252993367
f( u2 )
      -0.000006321773508011219
      -0.00001153324122649124
```

4. Recall the difference between Newton's method and the secant method for a single algebraic equation in a single variable. Suppose instead, you did not have the derivative. How would you generalize the secant method for two algebraic equations in two variables, or n algebraic equations in n variables? Recall that a plane is defined by three points.

As a plane is defined by three points, we could start with three initial points $\mathbf{x}_0$, $\mathbf{x}_1$ and $\mathbf{x}_2$ and then for the first function, f_1, we could find a tangent plane passing through $(\mathbf{x}_0, f_1(\mathbf{x}_0))$, $(\mathbf{x}_1, f_1(\mathbf{x}_1))$ $(\mathbf{x}_2, f_1(\mathbf{x}_2))$ and for the second function, f_2, we could find a similar tangent plane. Recall that a plane is a function of the form $\phi(\mathbf{x}) = ax_1 + bx_2 + c$, so to pass through three points, we require that

$$\begin{aligned} ax_{0,1} + bx_{0,2} + c &= f_1(\mathbf{x}_0) \\ ax_{1,1} + bx_{1,2} + c &= f_1(\mathbf{x}_1) \\ ax_{2,1} + bx_{2,2} + c &= f_1(\mathbf{x}_2) \end{aligned}$$

This is a system of three equations and three unknowns, and thus, in general, has a solution, and this defines a plane. We could do this for f_2, as well. With two secant planes, we can find a simultaneous solution or root, and hence that solution is $\mathbf{x}_3$.

5. Suppose you have the ordinary differential equation $y^{(1)}(t) = \sin(y(t))$ and you know that $y(0) = 1$ and $y(0.1) = 1.086355758991046$. Use a cubic spline to approximate $y(0.05)$.

First, calculating $\delta = (0.05 - 0)/(0.1 - 0) = 0.5$.

We need to solve the system of linear equations

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0.1\cdot\sin(1) \\ 1 & 1 & 1 & 1 & 1.086355758991046 \\ 3 & 2 & 1 & 0 & 0.1\cdot\sin(1.086355758991046) \end{array}\right)$$

Thus, we have

```
A = [0 0 0 1; 0 0 1 0; 1 1 1 1; 3 2 1 0]
      A = 0       0       0       1
          0       0       1       0
          1       1       1       1
          3       2       1       0

>> b = [1 0.1*sin(1) 1.086355758991046 0.1*sin(1.086355758991046)]'
      b = 1
          0.08414709848078966
          1.086355758991046
          0.08849356226254429

p = A \ b
      p = -0.00007085723875786790
           0.002279517749014116
           0.08414709848078966
           1

polyval( p, 0.5 )
      1.042634571522804
```

6. Suppse you knew that $y(a) = y_a$, $y(b) = y_b$, $y^{(1)}(a) = y_a^{(1)}$, $y^{(1)}(b) = y_b^{(1)}$, $y^{(2)}(a) = y_a^{(2)}$, $y^{(2)}(b) = y_b^{(2)}$. Write down the system of linear equations that would find the quintic (degree five) polynomial that satisfies these conditions.

The quintic polynomial is $p(t) = at^5 + bt^4 + ct^3 + dt^2 + et + f$, and its first and second derivatives are

$$p^{(1)}(t) = 5at^4 + 4bt^3 + 3ct^2 + 2dt + e$$

$$p^{(2)}(t) = 20at^3 + 12bt^2 + 6ct + 2d$$

Now, we scale and shift to 0 and 1 so that our algorithm can be re-used, Thus, the slopes will be modified by h and the concavities by h^2, so we have

$$p(0) = y_a,\ p^{(1)}(0) = hy_a^{(1)},\ p^{(2)}(0) = h^2y_a^{(2)},\ p(1) = y_a,\ p^{(1)}(1) = hy_a^{(1)},\ p^{(2)}(1) = h^2y_a^{(2)}$$

$$\left(\begin{array}{cccccc:c} 0 & 0 & 0 & 0 & 0 & 1 & y_a \\ 0 & 0 & 0 & 0 & 1 & 0 & hy_a^{(1)} \\ 0 & 0 & 0 & 2 & 0 & 0 & h^2y_a^{(2)} \\ 1 & 1 & 1 & 1 & 1 & 1 & y_b \\ 5 & 4 & 3 & 2 & 1 & 0 & hy_b^{(1)} \\ 20 & 12 & 6 & 2 & 0 & 0 & h^2y_b^{(2)} \end{array}\right)$$

7. Using Euler's method, approximate $y(1)$ with $h = 0.2$ and again with $h = 0.1$ for the initial-value problem defined by

$$y^{(1)}(t) = 2y(t) + t - 1$$
$$y(0) = 1$$

```
a = 0;
b = 1;
t0 = 0;
y0 = 1;
f = @(t, y)(2*y + t - 1);
h = 0.2;
N = 5;
ts = a:h:b;
ys = zeros( 1, N + 1 );
ys(1) = y0;
for k = 1:N
    ys(k + 1) = ys(k) + h*f(ts(k), ys(k));
end
ys
   ys = 1 1.2 1.52 2.008 2.7312 3.7837

h = 0.1;
N = 10;
tss = a:h:b;
yss = zeros( 1, N + 1 );
yss(1) = y0;
for k = 1:N
    yss(k + 1) = yss(k) + h*f(tss(k), yss(k));
end
yss
   yss = 1 1.1 1.23 1.396 1.6052 1.8662 2.1895 2.5874 3.0749 3.6698 4.3938
```

8. In Question 7, you approximated $y(0.2)$ with $h = 0.2$, and $y(0.1)$ with $h = 0.1$. The correct solutions to sixteen significant digits are $y(0.2) = 1.268868523230952$ and $y(0.1) = 1.116052068620128$. Show that the error of one step of Euler's method is $O(h^2)$ by showing that the error of your approximation at $t = 0.1$ is approximately one quarter the error at $t = 0.2$.

The approximations are 1.2 and 1.1, respectively, and thus the errors are 0.068868523230952 and 0.016052068620128, and we note that the second number is approximately one quarter the first.

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